Horn Clauses in Hybrid-Dynamic First-Order Logic

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Reconfiguration paradigm

- The present work is a part of a larger project: logical foundations of reconfiguration paradigm
- In many cases, the applications with reconfigurable features involve safety-critical areas. For example, the new generation of software-driven medical devices such as imaging machines, pill cameras, artificial pacemakers, the insulin infusion pump, etc.
- The safety requirements can be fulfilled only with formal methods
- The safety requirements can be fulfilled only by applying formal methods.
- “One of the main issues is that there is no real formal method of implementing the reconfiguration of an application” [Szepesi and Ciocarlie, Theory Appl. Math. Comput. Sci. 2011].
Reconfigurable systems can be regarded as transition systems in the following way:

- the configurations are states, and
- switching from one configuration to another is a transition.

**Hybrid dynamic logics** are modal logics that can describe transitions systems and express the dynamics of (re)configurations:

- the configurations of the software in use today may be modeled with first-order logic, higher-order logic, rewriting logic, etc; these are considered **base logics**;
- the construction of a hybrid logic on top of a base logic is called **hybridization** [Diaconescu and Madeira, Math. Struct. Comput. Sci. 2016].
Formal method design

- **Institution** category-based formalization of the intuitive definition of logical system

- **Hybrid institution** formalizes the notion of hybrid logic by supplementing the definition of institution with an additional structure to extract (a) nominals and modalities from signatures, and (b) frames from models.

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- the design of a formal method consists of several steps depicted in the left node of the figure above
- each step relies on some model-theoretic or proof-theoretic properties which should be defined at an appropriate abstraction level in the framework of hybrid institutions
- the proof of model-theoretic and proof-theoretic properties should be performed within the framework of hybrid institutions as depicted in the right side of the figure above
- once proved the abstract results are instantiated to concrete logical systems
Preparation status

- Abstract proof calculus for hybrid institutions whose sentences are Horn clauses [Gaina, Formal Asp. Comput. 2017]
- Proof calculus for the Horn clauses of hybrid-dynamic first-order logic [Gaina and Tutu, TABLEAUX 2019]
- Forcing in hybrid institutions [Gaina, Journal of the ACM accepted]

The following figure shows the advancements for this project and how this work will be completed in the present project.

<table>
<thead>
<tr>
<th>Proof calculi (for all sentences)</th>
<th>Hybrid institutions</th>
<th>Hybrid-dynamic institutions</th>
</tr>
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<tbody>
<tr>
<td>[Gaina, Journal of the ACM accepted]</td>
<td>future work</td>
<td></td>
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Semantics first: Kripke models

- mathematical structures used to model the behaviour of a system
- underlie the semantics of modal, hybrid, dynamic and temporal logics

- typically, a Kripke structure consists in a (hyper)graph where:
  - each node (possible world) represents a state of the system
  - each edge (part of an accessibility relation) represents a transition

- the original definition has been extended in various ways:
  - complex algebraic structures as labels of the states
  - structured actions as labels of transitions
  - model constraints
Hybrid-Dynamic First-Order Logic with user-defined sharing

A logic for specifying and reasoning about Kripke structures

... obtained by enriching first-order logic
  with features that are characteristic to
  hybrid logics (nominals and local-satisfaction operators) and
dynamic logics (structured actions over modalities)

... and with a number of distinctive attributes:

• a first-order structure on possible worlds
• supports sharing between possible worlds / rigidity constraints
• hybrid terms (annotated with nominals)
Signatures

**Definition.** An HDFOL-signature is a tuple $\Delta = (\Sigma^n, \Sigma^r \subseteq \Sigma)$, where:

- $\Sigma^n = (F^n, P^n)$ is a single-sorted first-order signature of nominals, where $F^n_i$ is a set of nominal operations of arity $i \in \mathbb{N}$
- $\Sigma^r = (S^r, F^r, P^r)$ is a many-sorted signature of rigid symbols, where $F^r_{ar \rightarrow s}$ is a set of operations of arity $ar \in S^r$ and sort $s \in S^r$
- $\Sigma = (S, F, P)$ is a many-sorted first-order signature of both rigid and flexible symbols

An HDFOL-signature morphism $\varphi: \Delta \rightarrow \Delta'$ consists of a pair of first-order signature morphisms $\varphi^n: \Sigma^n \rightarrow \Sigma'^n$ and $\varphi: \Sigma \rightarrow \Sigma'$ such that $\varphi(\Sigma^r) \subseteq \Sigma'^r$. 
Semantics

Definition. A Kripke model of $\Delta = (\Sigma^\text{n}, \Sigma^r \subseteq \Sigma)$ is a pair $\langle W, M \rangle$:

- $W$ is a $\Sigma^\text{n}$-model, whose carrier set we denote by $|W|$;
- $M = (M_w)_{w \in |W|}$ is a family of $\Sigma$-models, indexed by worlds, such that $M_{w_1, s} = M_{w_2, s}$ for all $w_1, w_2 \in |W|$ and symbols $s$ in $\Sigma^r$ (rigid symbols have the same interpretation across possible worlds).

A homomorphism $h: \langle V, N \rangle \rightarrow \langle W, M \rangle$ is also a pair, consisting of first-order homomorphisms $h: V \rightarrow W$ and $h_v: N_v \rightarrow M_{h(v)}$, for every world $v \in |V|$, such that $h_{v_1, s} = h_{v_2, s}$ for all $v_1, v_2 \in |V|$ and $s \in S^r$. 

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Hybrid-Dynamic First-Order Logic

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The set **actions** over $\Delta$ is defined in an inductive fashion, according to the following grammar:

$$a ::= \lambda \in P^n_2 \mid a \land a \mid a \lor a \mid a^*$$

The **atomic sentences** defined over a signature $\Delta$ are given by:

- **nominal equations and relations**
  $$\rho ::= k_1 = k_2 \mid \lambda(k') \mid t_1 =_{k,s} t_2 \mid \varpi(t) \mid \pi(k; t)$$

- **hybrid equations and relations**

**Full sentences** over $\Delta$ are built from atomic sentences according to the following grammar:

$$\gamma ::= \rho \mid a(k_1, k_2) \mid \@_k \gamma \mid \neg \gamma \mid \land \Gamma \mid \downarrow z \cdot \gamma' \mid \forall X \cdot \gamma''$$

where $z$ is a nominal variable, $X$ is a set of variables, and $\gamma'$, $\gamma''$ are sentences over the extended signatures $\Delta[z]$ and $\Delta[X]$, respectively.
Local-satisfaction relation

Definition. Given a Kripke model \( \langle W, M \rangle \) of a signature \( \Delta \) and a possible world \( w \in |W| \), we have, for atomic sentences:

- \( \langle W, M \rangle \models^w k_1 = k_2 \) iff \( W_{k_1} = W_{k_2} \)
- \( \langle W, M \rangle \models^w \lambda(k) \) iff \( W_k \in W_\lambda \)
- \( \langle W, M \rangle \models^w t_1 =_k t_2 \) iff \( \langle W, M \rangle_{t_1} = \langle W, M \rangle_{t_2} \)
- \( \langle W, M \rangle \models^w \varpi(t) \) iff \( \langle W, M \rangle_t \in M_{w,\varpi} \)
- \( \langle W, M \rangle \models^w \pi(k; t) \) iff \( \langle W, M \rangle_t \in M_{w',\pi} \), where \( w' = W_k \)
Local-satisfaction relation

**Definition.** Given a Kripke model \( \langle W, M \rangle \) of a signature \( \Delta \) and a possible world \( w \in |W| \), we have, for full sentences:

- \( \langle W, M \rangle \models^w a(k_1, k_2) \) iff \( (W_{k_1}, W_{k_2}) \in W_a \)
- \( \langle W, M \rangle \models^w @_k \gamma \) iff \( \langle W, M \rangle \models^{w'} \gamma \), where \( w' = W_k \)
- \( \langle W, M \rangle \models^w \neg \gamma \) iff \( \langle W, M \rangle \not\models^w \gamma \)
- \( \langle W, M \rangle \models^w \land \Gamma \) iff \( \langle W, M \rangle \models^w \gamma \) for all \( \gamma \in \Gamma \)
- \( \langle W, M \rangle \models^w \downarrow z \cdot \gamma \) iff \( \langle W, M \rangle \models^{z \leftarrow w} \gamma \)
- \( \langle W, M \rangle \models^w \forall X \cdot \gamma \) iff \( (W', M') \models^w \gamma \) for all \( \Delta[X]\)-expansions \( (W', M') \) of \( \langle W, M \rangle \)
Expressivity and relationship to other modal logics

Support for conventional modal operators

- $[a] \gamma \triangleq \downarrow z \cdot \forall z' \cdot a(z, z') \Rightarrow \@ z' \gamma$
- $\langle a \rangle \gamma \triangleq \downarrow z \cdot \exists z' \cdot a(z, z') \land \@ z' \gamma$

Support for (linear) temporal operators

- $\Diamond \gamma \triangleq \downarrow z \cdot \@_{\text{next}(z)} \gamma$
- $\rho \text{ Until } \gamma \triangleq \exists z \cdot \Diamond (z \land \gamma) \land \Box (\Diamond z \Rightarrow \rho)$

Hybrid (annotated) terms vs ordinary terms

- $c_1(k) = \sigma(k; c_2(k))$ is equivalent to $\@_k c_1 = \sigma(c_2)$
- $c_3(k) = \sigma(k; c_4(k_0))$ is equivalent to $\exists x \cdot \@_{k_0} x = c_4 \land \@_k c_3 = \sigma(x)$

Implicit vs explicit dependence on possible worlds

- the nominal sentence $k$ is equivalent to $\downarrow z \cdot z = k$
- the store sentence $\downarrow z \cdot \gamma$ is equivalent to $\exists z \cdot (z \land \gamma)$
Event-based transition system

Events:
- submit-id, submit-pswd, cancel
- try-again, hint

Attributes:
- status, uid, attpts, observations
- type-id, type-pswd, random values
Setting the stage for Birkhoff completeness

Goal: syntactic characterization of the satisfiability relations $\Gamma \models_\Delta \gamma$

$$\langle W, M \rangle \models \gamma$$ for all $\Delta$-models $\langle W, M \rangle$ such that $\langle W, M \rangle \models \Gamma$

• both $\Gamma$ and $\gamma$ belong to the Horn-clause fragment of HDFOL

Definition. By Horn clause, we mean a sentence obtained from atomic sentences by repeated applications of the following sentence-building operators, in any order:

• retrieve
• implication (hypothesis: only atoms or action relations)
• store
• universal quantification
• necessity, next
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A layered approach

• develop progressively a series of syntactic entailment relations

\[ \Gamma \vdash \gamma \]

• where each layer builds on the previous one to allow for more general antecedents or consequents

• three major steps / entailment relations

Atomic completeness: both \( \Gamma \) and \( \gamma \) are atomic
Quasi-completeness: \( \Gamma \) is arbitrary, but \( \gamma \) is atomic
Horn-clause completeness: both \( \Gamma \) and \( \gamma \) are arbitrary

• all are sound and complete; only the first two are also compact
• atomic completeness is much more difficult to establish than it may seem at first sight...
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• atomic completeness is much more difficult to establish than it may seem at first sight...
**Atomic completeness**

**Lemma.** For every set $\Gamma$ of nominal equations over a signature $\Delta$, there exists a reachable initial model $(W^\Gamma, M^\Gamma)$ such that

$$\Gamma \models \rho \iff (W^\Gamma, M^\Gamma) \models \rho \iff \Gamma \vdash \rho$$

for all nominal or hybrid equations $\rho$ over $\Delta$.

**Theorem (Atomic completeness).** Every set $\Gamma$ of atomic sentences over a signature $\Delta$ has a reachable initial model $(W^\Gamma, M^\Gamma)$ such that

$$\Gamma \models \rho \iff (W^\Gamma, M^\Gamma) \models \rho \iff \Gamma \vdash \rho$$

for all atomic sentences $\rho$ over $\Delta$. 
Quasi-completeness

Theorem (Quasi-completeness). Let:

- $\Gamma$ be a set of clauses over a signature $\Delta$,
- $\Gamma_0 = \{\rho \in \text{Sen}^{\text{HDCLS}}(\Delta) \mid \Gamma \vdash \rho \land \rho \text{ is atomic}\}$,
- $(W^{\Gamma_0}, M^{\Gamma_0})$ a reachable initial model of $\Gamma_0$ as before.

Then the following statements are equivalent:

1. $\Gamma \vdash \rho$
2. $(W^{\Gamma_0}, M^{\Gamma_0}) \models \rho$
3. $\Gamma \vdash \rho$
Horn-clause completeness

Proof rules

Theorem (Birkhoff completeness). The entailment relation generated by the rules presented thus far is sound and complete.

\[ \Gamma \vdash \gamma \text{ if and only if } \Gamma \vDash \gamma \]

Moreover, in the absence of (Star\textsubscript{I}), it is also compact.

Proposition (Lack of compactness). HDCLS does not admit an entailment relation that is sound, complete, and also compact.
Conclusions

To sum up

- we have introduced a hybrid-dynamic first-order logic
- seen how it relates to modal/temporal/hybrid logics
- presented a sound and complete calculus for its Horn fragment
- touched upon the limits of compactness

Ongoing work

- decidability properties and support for executable specifications
- case studies & a prototype implementation
Conclusions

To sum up
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Ongoing work
• decidability properties and support for executable specifications
• case studies & a prototype implementation
Thank you!
Atomic completeness

Rules for nominal terms

\((R^n)\) \[
\Gamma \vdash k = k
\]

\((S^n)\) \[
\Gamma \vdash k_1 = k_2 \\
\Gamma \vdash k_2 = k_1
\]

\((T^n)\) \[
\Gamma \vdash k_1 = k_2 \\
\Gamma \vdash k_2 = k_3 \\
\Gamma \vdash k_1 = k_3
\]

\((F^n)\) \[
\Gamma \vdash k_1 = k_2 \\
\Gamma \vdash o(k_1) = o(k_2)
\]

\((P^n)\) \[
\Gamma \vdash \lambda(k_1) \\
\Gamma \vdash k_1 = k_2 \\
\Gamma \vdash \lambda(k_2)
\]
Atomic completeness

Rules for sharing

\[(W^h)\]
\[
\Gamma \vdash k = k' \\
\Gamma \vdash t =_{k,s} \delta_{k'/k}(t)
\]
where \(s \in S^r\)

\[(W^r)\]
\[
\Gamma \vdash t_1 =_{k_1,s} t_2 \\
\Gamma \vdash t_1 =_{k_2,s} t_2
\]
where \(s \in S^r\)

\[(W^f)\]
\[
\Gamma \vdash k = k' \\
\Gamma \vdash t_1 =_{k} t_2 \\
\Gamma \vdash \delta_{k/k'}(t_1) =_{k'} \delta_{k/k'}(t_2)
\]
Atomic completeness

Rules for hybrid terms

(F') \[ \Gamma \vdash t_1 =_{k,ar} t_2 \]

(Γ) \[ \Gamma \vdash \sigma(t_1) =_{k,s} \sigma(t_2) \]

(P') \[ \Gamma \vdash t_1 =_k t_2 \quad \Gamma \vdash \pi(t_1) \]

(Γ) \[ \Gamma \vdash \pi(t_2) \]

(P'H) \[ \Gamma \vdash k_1 = k_2 \quad \Gamma \vdash \pi(k_1; t_1) \]

(Γ) \[ \Gamma \vdash \pi(k_2; \delta_{k_1/k_2}(t_1)) \]

(F) \[ \Gamma \vdash t_1 =_{k,ar} t_2 \]

(Γ) \[ \Gamma \vdash \sigma(k; t_1) =_{k,s} \sigma(k; t_2) \]

(PF) \[ \Gamma \vdash t_1 =_k t_2 \quad \Gamma \vdash \pi(k; t_1) \]

(Γ) \[ \Gamma \vdash \pi(k; t_2) \]

(PFH) \[ \Gamma \vdash k_1 = k_2 \quad \Gamma \vdash \pi(k_1; t_1) \]

(Γ) \[ \Gamma \vdash \pi(k_2; \delta_{k_1/k_2}(t_1)) \]

(Ret0) \[ \Gamma \vdash \@_k \rho \]

(Γ) \[ \Gamma \vdash \rho \]
Quasi-completeness

Rules for action relations

(Comp) \[ \frac{\Gamma \vdash a_1(k_1, k_2) \quad \Gamma \vdash a_2(k_2, k_3)}{\Gamma \vdash (a_1 \circ a_2)(k_1, k_3)} \]

(Union) \[ \frac{\Gamma \vdash a_i(k_1, k_2)}{\Gamma \vdash (a_1 \cup a_2)(k_1, k_2)} \]

(Refl) \[ \frac{\Gamma \vdash k_1 = k_2}{\Gamma \vdash a^*(k_1, k_2)} \]

(Star) \[ \Gamma \vdash a(k_i, k_{i+1}) \text{ for } 0 \leq i < n \]

\[ \Gamma \vdash a^*(k_0, k_n) \]
Quasi-completeness

Rules for Horn clauses

(Ret@) \[ \Gamma \vdash @_{k_1} @_{k_2} \gamma \]
\[ \Gamma \vdash @_{k_2} \gamma \]

(Ret1) \[ \Gamma \vdash \gamma \]
\[ \Gamma \vdash @_k \gamma \]

(ImpE) \[ \Gamma \vdash @_k (\land H \Rightarrow \gamma) \]
\[ \Gamma \cup H \vdash @_k \gamma \]

(StoreE) \[ \Gamma \vdash @_k \downarrow z \cdot \gamma \]
\[ \Gamma \vdash @_k \theta_{z \leftarrow k}(\gamma) \]

(Substq) \[ \Gamma \vdash @_k \forall X \cdot \gamma \]
\[ \Gamma \vdash @_k \theta(\gamma) \]
Horn-clause completeness

Additional rules for Horn clauses

\[
\begin{align*}
\text{(Ret}_E) & \quad \frac{\Gamma \vdash \Delta[z] \ @z \gamma}{\Gamma \vdash \Delta \gamma} \\
\text{(Store}_I) & \quad \frac{\Gamma \vdash @k \theta_{z \leftarrow k}(\gamma)}{@k \downarrow z \cdot \gamma} \\
\text{(Imp}_I) & \quad \frac{\Gamma \vdash H \vdash @k \gamma}{\Gamma \vdash \Delta \gamma} \\
\text{(Quant}_I) & \quad \frac{\Gamma \vdash \Delta[X] \ @k \gamma}{\Gamma \vdash \Delta \ @k \forall X \cdot \gamma}
\end{align*}
\]
Horn-clause completeness

Additional rules for action relations

(Comp_1) \[ E \cup \{a_1(k_1, z), a_2(z, k_2)\} \vdash_{\Delta[z]} e \]
\[ E \cup \{(a_1 \# a_2)(k_1, k_2)\} \vdash_{\Delta} e \]

(Union_1) \[ E \cup \{a_i(k_1, k_2)\} \vdash e \text{ for } i \in \{1, 2\} \]
\[ E \cup \{(a_1 \cup a_2)(k_1, k_2)\} \vdash e \]

(Star_1) \[ E \cup \{a^n(k_1, k_2)\} \vdash e \text{ for all } n \in \mathbb{N} \]
\[ E \cup \{a^*(k_1, k_2)\} \vdash e \]
Generic insulin infusion pump

Pump controller - an abstract representation of generic insulin pump software

- **Main functionality**: command the pump delivery mechanism to propel insulin stored in the drug reservoir to the patient through the drug delivery interface and the infusion set

- **Overall responsibility**: ensure correct operation of the model

More about pump controller functionality:

- interacts with the patient through a user interface
- it alerts the patient when abnormal conditions arise
- recommends appropriate bolus dosages with the help of a bolus calculator and a food database
- manages and checks parameters and programs related to insulin administration;
- logs important data and events during pump use to facilitate clinical use analysis and problem diagnosis